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## Mining Data Streams (Part 2)

Mining of Massive Datasets
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## Today's Lecture

- More algorithms for streams:
- (1) Filtering a data stream: Bloom filters
- Select elements with property $\mathbf{x}$ from stream
- (2) Counting distinct elements: Flajolet-Martin
- Number of distinct elements in the last $\boldsymbol{k}$ elements of the stream
- (3) Estimating moments: AMS method
- Estimate std. dev. of last $\boldsymbol{k}$ elements
- (4) Counting frequent items
(1) Filtering Data Streams


## Filtering Data Streams

- Each element of data stream is a tuple
- Given a list of keys S
- Determine which tuples of stream are in $S$
- Obvious solution: Hash table
- But suppose we do not have enough memory to store all of $\boldsymbol{S}$ in a hash table
- E.g., we might be processing millions of filters on the same stream


## Applications

- Example: Email spam filtering
- We know 1 billion "good" email addresses
- If an email comes from one of these, it is NOT spam
- Publish-subscribe systems
- You are collecting lots of messages (news articles)
- People express interest in certain sets of keywords
- Determine whether each message matches user's interest


## First Cut Solution (1)

- Given a set of keys $S$ that we want to filter
- Create a bit array B of $n$ bits, initially all Os
- Choose a hash function $h$ with range $[0, n)$
- Hash each member of $s \in S$ to one of $n$ buckets, and set that bit to 1, i.e., $B[h(s)]=1$
- Hash each element $a$ of the stream and output only those that hash to bit that was set to 1
- Output $\boldsymbol{a}$ if $\mathrm{B}[\mathrm{h}(\mathrm{a})]==1$


## First Cut Solution (2)



Drop the item.
It hashes to a bucket set
to $\mathbf{O}$ so it is surely not in $\boldsymbol{S}$.

- Creates false positives but no false negatives
- If the item is in $S$ we surely output it, if not we may still output it


## First Cut Solution (3)

- |S| = 1 billion email addresses
$|B|=1 G B=8$ billion bits
- If the email address is in $S$, then it surely hashes to a bucket that has the big set to 1, so it always gets through (no false negatives)
- Approximately $1 / 8$ of the bits are set to 1 , so about $1 / 8^{\text {th }}$ of the addresses not in $S$ get through to the output (false positives)
- Actually, less than $1 / \mathbf{8}^{\text {th }}$, because more than one address might hash to the same bit


## Analysis: Throwing Darts (1)

- More accurate analysis for the number of false positives
- Consider: If we throw $\boldsymbol{m}$ darts into $\boldsymbol{n}$ equally likely targets, what is the probability that a target gets at least one dart?
- In our case:
- Targets = bits/buckets
- Darts = hash values of items


## Analysis: Throwing Darts (2)

- We have $\boldsymbol{m}$ darts, $\boldsymbol{n}$ targets
- What is the probability that a target gets at least one dart?



## Analysis: Throwing Darts (3)

- Fraction of 1s in the array B =
$=$ probability of false positive $=1-e^{-m / n}$
- Example: $\mathbf{1 0}^{9}$ darts, $\mathbf{8 \cdot 1 0 ^ { 9 }}$ targets
- Fraction of $\mathbf{1 s}$ in $\mathbf{B}=\mathbf{1 - \mathrm { e } ^ { - 1 / 8 }}=\mathbf{0 . 1 1 7 5}$
- Compare with our earlier estimate: 1/8=0.125


## Bloom Filter

- Consider: $|\mathbf{S}|=\boldsymbol{m},|\mathbf{B}|=n$
- Use $\boldsymbol{k}$ independent hash functions $\boldsymbol{h}_{1}, \ldots, \boldsymbol{h}_{\boldsymbol{k}}$
- Initialization:
- Set B to all Os
- Hash each element $\boldsymbol{s} \in \boldsymbol{S}$ using each hash function $\boldsymbol{h}_{\boldsymbol{i}}$, set $\mathrm{B}\left[h_{i}(s)\right]=1 \quad$ (for each $\left.\boldsymbol{i}=\mathbf{1}, . ., \boldsymbol{k}\right)$ (note: we have a single array B!)
- Run-time:
- When a stream element with key $\boldsymbol{x}$ arrives
- If $\mathrm{B}\left[h_{i}(x)\right]=1$ for all $i=1, \ldots, k$ then declare that $x$ is in $S$
- That is, $\boldsymbol{x}$ hashes to a bucket set to $\mathbf{1}$ for every hash function $\boldsymbol{h}_{i}(\boldsymbol{x})$
- Otherwise discard the element $\boldsymbol{x}$


## Bloom Filter -- Analysis

- What fraction of the bit vector B are 1s?
- Throwing $\boldsymbol{k} \cdot \boldsymbol{m}$ darts at $\boldsymbol{n}$ targets
- So fraction of 1 s is ( $\left.1-e^{-k m / n}\right)$
- But we have $\boldsymbol{k}$ independent hash functions and we only let the element $\boldsymbol{x}$ through if all $\boldsymbol{k}$ hash element $\boldsymbol{x}$ to a bucket of value 1
- So, false positive probability $=\left(1-e^{-k m / n}\right)^{k}$


## Bloom Filter - Analysis (2)

- $m=1$ billion, $n=8$ billion
- $\mathbf{k}=\mathbf{1}:\left(1-\mathrm{e}^{-1 / 8}\right)=0.1175$
- $\mathbf{k}=\mathbf{2}:\left(1-e^{-1 / 4}\right)^{2}=0.0493$
- What happens as we keep increasing $k$ ?

- "Optimal" value of $k: n / m \ln (2)$
- In our case: Optimal k=8 $\ln (2)=5.54 \approx 6$
- Error at $k=6:\left(1-e^{-1 / 6}\right)^{2}=0.0235$


## Bloom Filter: Wrap-up

- Bloom filters guarantee no false negatives, and use limited memory
- Great for pre-processing before more expensive checks
- Suitable for hardware implementation
- Hash function computations can be parallelized
- Is it better to have $\mathbf{1}$ big $\mathbf{B}$ or $\boldsymbol{k}$ small Bs?
- It is the same: $\left(1-e^{-k m / n}\right)^{k}$ vs. $\left(1-e^{-m /(n / k)}\right)^{k}$
- But keeping $\mathbf{1}$ big $\mathbf{B}$ is simpler
(2) Counting Distinct Elements


## Counting Distinct Elements

- Problem:
- Data stream consists of a universe of elements chosen from a set of size $\mathbf{N}$
- Maintain a count of the number of distinct elements seen so far
- Obvious approach: Maintain the set of elements seen so far
- That is, keep a hash table of all the distinct elements seen so far


## Applications

- How many different words are found among the Web pages being crawled at a site?
- Unusually low or high numbers could indicate artificial pages (spam?)
- How many different Web pages does each customer request in a week?
- How many distinct products have we sold in the last week?


## Using Small Storage

- Real problem: What if we do not have space to maintain the set of elements seen so far?
- Estimate the count in an unbiased way
- Accept that the count may have a little error, but limit the probability that the error is large


## Flajolet-Martin Approach

- Pick a hash function $\boldsymbol{h}$ that maps each of the $\mathbf{N}$ elements to at least $\log _{2} \mathbf{N}$ bits
- For each stream element $\boldsymbol{a}$, let $\boldsymbol{r}(\boldsymbol{a})$ be the number of trailing $\mathbf{O s}$ in $\boldsymbol{h ( a )}$
- $r(a)=$ position of first 1 counting from the right
- E.g., say $h(a)=12$, then 12 is 1100 in binary, so $r(a)=2$
- Record $R=$ the maximum $r(a)$ seen
- $\mathbf{R}=\boldsymbol{m a x}_{\mathrm{a}} \mathbf{r}(\mathbf{a})$, over all the items $\boldsymbol{a}$ seen so far
- Estimated number of distinct elements $=2^{R}$


## Why It Works: Intuition

- Very very rough and heuristic intuition why Flajolet-Martin works:
- $\boldsymbol{h}(\boldsymbol{a})$ hashes $\boldsymbol{a}$ with equal prob. to any of $\boldsymbol{N}$ values
- Then $\boldsymbol{h}(a)$ is a sequence of $\log _{2} \mathbf{N}$ bits, where $2^{-r}$ fraction of all as have a tail of $r$ zeros
- About 50\% of as hash to ***0
- About 25\% of as hash to **00
- So, if we saw the longest tail of $\boldsymbol{r = 2}$ (i.e., item hash ending *100) then we have probably seen about 4 distinct items so far
- So, it takes to hash about $2^{r}$ items before we see one with zero-suffix of length $r$


## Why It Works: More formally

- Now we show why Flajolet-Martin works
- Formally, we will show that probability of finding a tail of $r$ zeros:
- Goes to 1 if $m>2^{r}$
- Goes to 0 if $m \ll 2^{r}$
where $m$ is the number of distinct elements seen so far in the stream
- Thus, $2^{R}$ will almost always be around $m$ !


## Why It Works: More formally

- What is the probability that a given $h(a)$ ends in at least $r$ zeros is $2^{-r}$
- $h(a)$ hashes elements uniformly at random
- Probability that a random number ends in at least $\boldsymbol{r}$ zeros is $\mathbf{2}^{-r}$
- Then, the probability of NOT seeing a tail of length $r$ among $m$ elements:



## Why It Works: More formally

- Note: $\left(1-2^{-r}\right)^{m}=\left(1-2^{-r}\right)^{2^{r}\left(m 2^{-r}\right)} \approx e^{-m 2^{-r}}$
- Prob. of NOT finding a tail of length $r$ is:
- If $m \ll 2^{r}$, then prob. tends to 1
- $\left(1-2^{-r}\right)^{m} \approx e^{-m 2^{-r}}=1$ as $\mathbf{m} / \mathbf{2}^{r} \rightarrow \mathbf{0}$
- So, the probability of finding a tail of length $r$ tends to 0
- If $m \gg 2^{r}$, then prob. tends to 0
- $\left(1-2^{-r}\right)^{m} \approx e^{-m 2^{-r}}=0 \quad$ as $\mathbf{m} / \mathbf{2}^{r} \rightarrow \infty$
- So, the probability of finding a tail of length $r$ tends to 1
- Thus, $2^{R}$ will almost always be around $m$ !


## Why It Doesn't Work

- $\mathrm{E}\left[2^{R}\right]$ is actually infinite
- Probability halves when $\boldsymbol{R} \rightarrow \boldsymbol{R}+\mathbf{1}$, but value doubles
- Workaround involves using many hash functions $h_{i}$ and getting many samples of $R_{i}$
- How are samples $R_{i}$ combined?
- Average? What if one very large value $\mathbf{2}^{R_{i}}$ ?
- Median? All estimates are a power of 2
- Solution:
- Partition your samples into small groups
- Take the median of groups
- Then take the average of the medians
(3) Computing Moments


## Generalization: Moments

- Suppose a stream has elements chosen from a set $A$ of $N$ values
- Let $m_{i}$ be the number of times value $i$ occurs in the stream
- The $\boldsymbol{k}^{\text {th }}$ moment is

$$
\sum_{i \in A}\left(m_{i}\right)^{k}
$$

## Special Cases

$$
\sum_{i \in A}\left(m_{i}\right)^{k}
$$

- $\mathbf{0}^{\text {th }}$ moment $=$ number of distinct elements
- The problem just considered
- $1^{\text {st }}$ moment $=$ count of the numbers of
elements = length of the stream
- Easy to compute
- $2^{\text {nd }}$ moment $=$ surprise number $S=$
a measure of how uneven the distribution is


## Example: Surprise Number

- Stream of length 100
- 11 distinct values
- Item counts: 10, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9 Surprise S = 910
- Item counts: 90, 1, 1, 1, 1, 1, 1, 1 ,1, 1, 1 Surprise S = 8,110


## AMS Method

- AMS method works for all moments
- Gives an unbiased estimate
- We will just concentrate on the $2^{\text {nd }}$ moment $S$
- We pick and keep track of many variables $X$ :
- For each variable $\boldsymbol{X}$ we store $\boldsymbol{X} . \boldsymbol{e l}$ and $\boldsymbol{X} . \boldsymbol{v a l}$
- X.el corresponds to the item $\boldsymbol{i}$
- X.val corresponds to the count of item $\boldsymbol{i}$
- Note this requires a count in main memory, so number of $X$ s is limited
- Our goal is to compute $S=\sum_{i} m_{i}^{2}$


## One Random Variable (X)

- How to set X.val and X.el?
- Assume stream has length $n$ (we relax this later)
- Pick some random time $\boldsymbol{t}(\boldsymbol{t}<\boldsymbol{n})$ to start, so that any time is equally likely
- Let at time $\boldsymbol{t}$ the stream have item $\boldsymbol{i}$. We set X.el = i
- Then we maintain count $\boldsymbol{c}(X . v a l=c)$ of the number of is in the stream starting from the chosen time $t$
- Then the estimate of the $2^{\text {nd }}$ moment $\left(\sum_{i} \boldsymbol{m}_{i}^{2}\right)$ is:

$$
S=f(X)=n(2 \cdot c-1)
$$

- Note, we will keep track of multiple $\mathbf{X}$ s, ( $\left.\mathbf{X}_{\mathbf{1}}, \mathbf{X}_{\mathbf{2}}, \ldots \mathbf{X}_{\mathbf{k}}\right)$ and our final estimate will be $S=1 / k \sum_{j}^{k} f\left(X_{j}\right)$


## Expectation Analysis



- $2^{\text {nd }}$ moment is $S=\sum_{i} \boldsymbol{m}_{i}^{2}$
- $c_{t} \ldots$ number of times item at time $t$ appears from time $t$ onwards ( $c_{1}=m_{a}, c_{2}=m_{a}-1, c_{3}=m_{b}$ )
- $E[f(X)]=\frac{1}{n} \sum_{t=1}^{n} n\left(2 c_{t}-1\right) \quad$ m, tolal count of
$=\frac{1}{n} \sum_{n}\left(1+3+5+\cdots+2 m_{i}-1\right) \underset{\substack{\text { item in the stieam } \\ \text { (we are assuming }}}{\substack{\text { in }}}$ (we are assuming
ream has length $\mathbf{n}$ )

Group times
by the value seen

Time t when the last $i$ is $\operatorname{seen}\left(c_{t}=1\right)$

> Time $t$ when the penultimate $i$ is seen $\left(c_{t}=2\right)$

Time $\boldsymbol{t}$ when the first $i$ is
$\operatorname{seen}\left(c_{t}=m_{i}\right)$

## Expectation Analysis



- $E[f(X)]=\frac{1}{n} \sum_{i} n\left(1+3+5+\cdots+2 m_{i}-1\right)$
- Little side calculation: $\left(1+3+5+\cdots+2 m_{i}-1\right)=$ $\sum_{i=1}^{m_{i}}(2 i-1)=2 \frac{m_{i}\left(m_{i}+1\right)}{2}-m_{i}=\left(m_{i}\right)^{2}$
Then $E[f(X)]=\frac{1}{n} \sum_{i} n\left(m_{i}\right)^{2}$
- So, $\mathrm{E}[\mathrm{f}(\mathrm{X})]=\sum_{i}\left(\boldsymbol{m}_{\boldsymbol{i}}\right)^{2}=S$
- We have the second moment (in expectation)!


## Higher-Order Moments

- For estimating $\mathbf{k}^{\text {th }}$ moment we essentially use the same algorithm but change the estimate:
- For $\mathbf{k}=\mathbf{2}$ we used $n(2 \cdot c-1)$
- For $\mathbf{k}=3$ we use: $n\left(3 \cdot c^{2}-3 c+1\right) \quad$ (where $\left.\mathbf{c}=X . v a l\right)$
- Why?
- For k=2: Remember we had $\left(1+3+5+\cdots+2 m_{i}-1\right)$ and we showed terms $\mathbf{2 c - 1}$ (for $\mathbf{c = 1}, \ldots, \mathbf{m}$ ) sum to $\boldsymbol{m}^{\mathbf{2}}$
- $\sum_{c=1}^{m} 2 c-1=\sum_{c=1}^{m} c^{2}-\sum_{c=1}^{m}(c-1)^{2}=m^{2}$
- So: $2 c-1=c^{2}-(c-1)^{2}$
- For $k=3: \mathbf{c}^{3}-(c-1)^{3}=3 c^{2}-3 c+1$
- Generally: Estimate $=n\left(c^{k}-(c-1)^{k}\right)$


## Combining Samples

- In practice:
- Compute $f(X)=n(2 c-1)$ for as many variables $\boldsymbol{X}$ as you can fit in memory
- Average them in groups
- Take median of averages
- Problem: Streams never end
- We assumed there was a number $n$, the number of positions in the stream
- But real streams go on forever, so $\boldsymbol{n}$ is a variable - the number of inputs seen so far


## Streams Never End: Fixups

(1) The variables $X$ have $n$ as a factor keep $\boldsymbol{n}$ separately; just hold the count in $\boldsymbol{X}$

- (2) Suppose we can only store $\boldsymbol{k}$ counts. We must throw some $\boldsymbol{X}$ s out as time goes on:
- Objective: Each starting time $\boldsymbol{t}$ is selected with probability $k / n$
- Solution: (fixed-size sampling!)
- Choose the first $\boldsymbol{k}$ times for $\boldsymbol{k}$ variables
- When the $\boldsymbol{n}^{\text {th }}$ element arrives ( $\boldsymbol{n} \boldsymbol{>} \boldsymbol{k}$ ), choose it with probability $\boldsymbol{k} / \boldsymbol{n}$
- If you choose it, throw one of the previously stored variables $\mathbf{X}$ out, with equal probability

Counting Itemsets

## Counting Itemsets

- New Problem: Given a stream, which items appear more than $s$ times in the window?
- Possible solution: Think of the stream of baskets as one binary stream per item
- 1 = item present; 0 = not present
- Use DGIM to estimate counts of 1 s for all items


010011100010100100010110110111001010110011010

## Extensions

- In principle, you could count frequent pairs or even larger sets the same way
- One stream per itemset
- Drawbacks:
- Only approximate
- Number of itemsets is way too big


## Exponentially Decaying Windows

- Exponentially decaying windows: A heuristic for selecting likely frequent item(sets)
" What are "currently" most popular movies?
- Instead of computing the raw count in last $\boldsymbol{N}$ elements
- Compute a smooth aggregation over the whole stream
- If stream is $\boldsymbol{a}_{\mathbf{1}}, \boldsymbol{a}_{\mathbf{2}}, \ldots$ and we are taking the sum of the stream, take the answer at time $t$ to be: $=\sum_{i=1}^{t} a_{i}(1-c)^{t-i}$
- c is a constant, presumably tiny, like $\mathbf{1 0}^{-6}$ or $\mathbf{1 0}^{-9}$
- When new $a_{t+1}$ arrives: Multiply current sum by (1-c) and add $\mathrm{a}_{\mathrm{t}+1}$


## Example: Counting Items

- If each $\boldsymbol{a}_{\boldsymbol{i}}$ is an "item" we can compute the characteristic function of each possible item $\boldsymbol{x}$ as an Exponentially Decaying Window
- That is: $\sum_{i=1}^{t} \boldsymbol{\delta}_{i} \cdot(\mathbf{1}-\boldsymbol{c})^{t-i}$
where $\boldsymbol{\delta}_{\mathbf{i}}=\mathbf{1}$ if $\mathrm{a}_{\mathrm{i}}=\mathbf{x}$, and $\mathbf{0}$ otherwise
- Imagine that for each item $\boldsymbol{x}$ we have a binary stream ( $\mathbf{1}$ if $\boldsymbol{x}$ appears, $\mathbf{0}$ if $\boldsymbol{x}$ does not appear)
- New item $\boldsymbol{x}$ arrives:
- Multiply all counts by (1-c)
- Add +1 to count for element $\boldsymbol{x}$
- Call this sum the "weight" of item $x$


## Sliding Versus Decaying Windows



- Important property: Sum over all weights $\sum_{t}(1-c)^{t}$ is $1 /[1-(1-c)]=1 / c$


## Example: Counting Items

- What are "currently" most popular movies?
- Suppose we want to find movies of weight > $1 / 2$
- Important property: Sum over all weights

$$
\sum_{t}(1-c)^{t} \text { is } 1 /[1-(1-c)]=1 / c
$$

- Thus:
- There cannot be more than $\mathbf{2 / c}$ movies with weight of $1 / 2$ or more
- So, $2 / c$ is a limit on the number of movies being counted at any time


## Extension to Itemsets

- Count (some) itemsets in an E.D.W.
" What are currently "hot" itemsets?
- Problem: Too many itemsets to keep counts of all of them in memory
- When a basket B comes in:
- Multiply all counts by (1-c)
- For uncounted items in B, create new count
- Add 1 to count of any item in B and to any itemset contained in $\mathbf{B}$ that is already being counted
- Drop counts < $1 / 2$
- Initiate new counts (next slide)


## Initiation of New Counts

- Start a count for an itemset $S \subseteq B$ if every proper subset of $\boldsymbol{S}$ had a count prior to arrival of basket B
- Intuitively: If all subsets of $\mathbf{S}$ are being counted this means they are "frequent/hot" and thus $S$ has a potential to be "hot"
- Example:
- Start counting $\boldsymbol{S}=\{\mathbf{i}, \mathbf{j}\} \mathrm{iff}$ both $\mathbf{i}$ and $\mathbf{j}$ were counted prior to seeing $\boldsymbol{B}$
- Start counting $\boldsymbol{S}=\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ iff $\{\mathbf{i}, \mathbf{j}\},\{i, k\}$, and $\{\mathbf{j}, \mathbf{k}\}$ were all counted prior to seeing $B$


## How many counts do we need?

- Counts for single items < (2/c)•(avg. number of items in a basket)
- Counts for larger itemsets = ??
- But we are conservative about starting counts of large sets
- If we counted every set we saw, one basket of $\mathbf{2 0}$ items would initiate $\mathbf{1 M}$ counts

